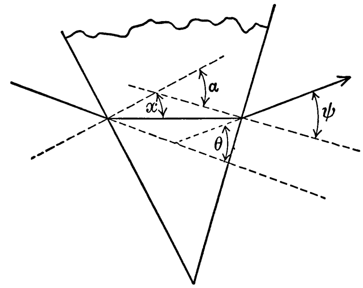
ϵ being the index of refraction. If θ is the angle of deviation, we have, by geometry, $\theta = \psi + \varphi - \alpha$ or

$$\theta = \sin^{-1} \epsilon \sin (\alpha - x) + \sin^{-1} \epsilon \sin x - \alpha.$$



For θ to be a minimum

$$\frac{d\theta}{dx} = \frac{-\epsilon \cos (\alpha - x)}{\sqrt{1 - \epsilon^2 \sin^2 (\alpha - x)}} + \frac{\epsilon \cos x}{\sqrt{1 - \epsilon^2 \sin^2 x}} = 0;$$

from which $x = \frac{1}{2}\alpha$ and therefore $\varphi = \psi = \sin^{-1} \epsilon \sin \frac{1}{2}\alpha$. That is, the angle of deflection or deviation is a minimum when the ray in the glass makes equal angles with the faces of the prism.

357 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University, South Bethlehem, Pa.

Two beads each of mass m connected by a string of length $2l$ and carrying a mass m' at its middle point are threaded symmetrically with respect to the major axis which is vertical on a smooth ellipse of eccentricity e and latus rectum $2l$. The string is held taut and horizontal, then released; find the velocities of the beads when the end ones impinge.

Solution by the Proposer.

There are two cases: I, when the end beads are at the extremities of the upper latus rectum II, when the end beads are at the extremities of the lower latus rectum. In either case when the end beads impinge their velocities will be equal and the velocity of m' will be zero. If a is the semi-major axis we have by the principle of work,

Case I

$$m'g\{l - a(1 - e)\} - 2mga(1 - e) = \frac{2m}{2} v^2;$$

or since

$$l = a(1 - e^2),$$

$$mv^2 = ag(1 - e)(m'e - 2m);$$

or

$$v^2 = \frac{gl}{m(1 + e)} \{m'e - 2m\}.$$

For the beads to impinge in this case $m' > 2m/e$.

Case II

$$m'g\{l + a(1 - e)\} + 2mga(1 - e) = \frac{2m}{2} v^2,$$

whence as before

$$v^2 = \frac{gl}{m(1 + e)} \{(2 + e)m' + 2m\}.$$

260 (Number Theory). Proposed by ALBERT A. BENNETT, University of Texas.

Let $\binom{n}{r}$ denote, as usual, the binomial coefficient $n!/[r!(n - r)!]$, where $\binom{n}{0} = 1$, but where n, r, $(n - r)$ are always to be supposed to be positive integers or zero. Let us define $k_i(m, n)$ as $\Sigma_i \binom{m-i+j}{i-j}\binom{n-j}{j}$. Prove that the following recursion formula is consistent:

$$\Sigma(-1)^i k_i(m, n) C_{m+n-i} = \binom{m + n}{m}$$

and determine $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, C_7 = 429, C_8 = 1{,}430$, etc. Prove also that these quantities satisfy the following relations, as well:

$$\sum_i (-1)^i C_{m-n-i} \binom{m - 1}{i} = 0$$

for each n where $2n \leq m$.

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

Consider the sum $\Sigma_i \Sigma_m \Sigma_n (-1)^i k_i(m, n) x^m y^n z^i$, where x, y, z are small enough to ensure absolute convergence. It equals $\Sigma_i \Sigma_j \Sigma_m \Sigma_n (-1)^i \binom{m-i+j}{i-j}\binom{n-j}{j} x^m y^n z^i$. It is not difficult to sum this in the order of writing, beginning with n. The result is that $(-1)^i k_i(m, n)$ is the coefficient of $x^m y^n z^i$ in $(1-x+x^2z)^{-1}(1-y+y^2z)^{-1}$, as may also be directly verified. It will then be sufficient to show that a power series $F(z) = C_0 + C_1 z + C_2 z^2 + \cdots$, independent of x and y can be found so that the coefficient of $x^m y^n z^{m+n}$ in $F(z)\cdot(1-x+x^2z)^{-1}(1-y+y^2z)^{-1}$ equals the corresponding coefficient in $\Sigma_m \Sigma_n \binom{m+n}{m} x^m y^n z^{m+n}$, that is, in $(1-xz-yz)^{-1}$. This does not mean that these generating functions are to be identical, since the first contains terms not of the form $Ax^m y^n z^{m+n}$. To escape this difficulty, write $x = u/z$, $y = v/z$. Then we have to find $F(z)$ so that the part independent of z in $F(z)\cdot(1-u/z+u^2/z)^{-1}(1-v/z+v^2/z)^{-1}$ is identical with $(1-u-v)^{-1}$, remembering that we are now to have positive and negative powers of z in the expansion of the former.

But

$$(1-u/z+u^2/z)^{-1}(1-v/z+v^2/z)^{-1} = (u-v)^{-1}(1-u-v)^{-1}\{(1-u/z+u^2/z)^{-1} - (1-v/z+v^2/z)^{-1}\};$$

so that, on expanding this in negative powers of z, and multiplying by $F(z)$, we find that $\Sigma_r C_r\{(u-u^2)^{r+1} - (v-v^2)^{r+1}\}$ must be identical with $u - v$. This can be satisfied only by having

$$\sum_r C_r(u-u^2)^{r+1} = u + A,$$

that is by making $zF(z) - A$ the expansion of u in ascending powers of z where $u^2 - u + z = 0$. Therefore,

$$zF(z) = A + \tfrac{1}{2} \pm \tfrac{1}{2}(1-4z)^{1/2};$$

whence,

$$A = -\tfrac{1}{2} \mp \tfrac{1}{2},$$

and

$$F(z) = \pm\{1-(1-4z)^{1/2}\}/(2z);$$

and the upper sign must be taken, since that alone makes $u + v < 1$, which is necessary for the convergence of $(1-u-v)^{-1}$. It follows that $C_r = 2(2r-1)!/[(r-1)!(r+1)!]$ when $r > 0$, and $C_0 = 1$. The numerical values may then be found, as stated.

As regards the second part of the problem, we observe that $\Sigma_i (-1)^i \binom{m-i}{i} C_{m-n-i}$ is the coefficient of $x^m z^{m-n}$ in the expansion of $F(z)\cdot\{1-(x-zx^2)\}^{-1}$ for small x and z; and by using partial fractions in terms of x, we find that this coefficient is the coefficient of z^{m-n} in

$$\{1-(1-4z)^{1/2}\}(2z)^{-1}\cdot[\{\tfrac{1}{2}+\tfrac{1}{2}(1-4z)^{1/2}\}^{m+1} - \{\tfrac{1}{2}-\tfrac{1}{2}(1-4z)^{1/2}\}^{m+1}](1-4z)^{-1/2}.$$

The second of the terms in square brackets has no powers of z below z^{m+1}, and may be omitted. The remaining part reduces to $\{\tfrac{1}{2}+\tfrac{1}{2}(1-4z)^{1/2}\}^m(1-4z)^{-1/2}$. For a similar reason, for powers of z below z^m, this may be replaced by

$$[\{\tfrac{1}{2}+\tfrac{1}{2}(1-4z)^{1/2}\}^m - \{\tfrac{1}{2}-\tfrac{1}{2}(1-4z)^{1/2}\}^m](1-4z)^{-1/2} = 2^{-m+1}\left\{\binom{m}{1}+\binom{m}{3}(1-4z)+\binom{m}{5}(1-4z)^2+\cdots\right\}$$

which has a degree less than $m/2$. The coefficient considered therefore vanishes when $0 < 2n \leqq m$. It is equal to 1 when $n = 0$, since

$$\begin{aligned}\sum_i(-1)^i\binom{m-i}{i}C_{m-i} &= \sum_i(-1)^i\binom{m-1-i}{i}C_{m-1-i} + \sum_i(-1)^i\binom{m+1-i}{i}C_{m-i}\\ &= \sum_i(-1)^i\binom{m-1-i}{i}C_{m-1-i} = \sum_i(-1)^i\binom{m-2-i}{i}C_{m-2-i} = \cdots\\ &= \sum_i(-1)^i\binom{1-i}{i}C_{1-i} = 1.\end{aligned}$$